1 Mathematics 122 Fall 2001

1.1 **Exam 2**

October 18, 2001 **Mr. KEY**

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Name

Technology used: _______________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a table or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. (15 points) A sphere of radius R can be obtained by rotating half of the circle $x^2+y^2=R^2$ around a coordinate axis. Use an integral for computing surface area to show that area is $A=4\pi R^2$.

   - **Solution when rotating about the $x-$ axis:**
     (a) $y=f(x)=\sqrt{R^2-x^2}$ so $f'(x)=\frac{1}{2}\left(R^2-x^2\right)^{-\frac{1}{2}}(-2x)=\frac{-x}{\sqrt{R^2-x^2}}$.
     (b) So $ds=\sqrt{1+(f'(x))^2}dx=\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}dx=\sqrt{1+\frac{x^2}{R^2-x^2}}dx=\sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}dx=\frac{R}{\sqrt{R^2-x^2}}dx$
     (c) Thus, the surface area is

$$\begin{aligned} SA &= 2\pi\int_{-R}^{R} f(x)\,ds \\ &= 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\frac{R}{\sqrt{R^2-x^2}}dx \\ &= 2\pi\int_{-R}^{R} R\,dx \\ &= 2\pi Rx|_{-R}^{R} \\ &= 2\pi R\left(R-(-R)\right) \\ &= 4\pi R^2 \end{aligned}$$

   - **Solution when rotating about the y axis:**
     (a) We have already computed $ds=\frac{R}{\sqrt{R^2-x^2}}dx$ so one-half of the surface area (the top half) is given by

$$\begin{aligned} SA &= 2\pi\int_{0}^{R} x\,ds \\ &= 2\pi\int_{0}^{R} x\,\frac{R}{\sqrt{R^2-x^2}}dx \end{aligned}$$

     (b) Now make the rule-of-thumb substitution

$$\begin{aligned} u &= R^2-x^2 \\ du &= (0-2x)\,dx \end{aligned}$$

to obtain

$$\begin{aligned}
SA &= 2\pi R \int_{u=R^2}^{0} \left(\frac{-1}{2}\right)\left(R^2 - x^2\right)^{-\frac{1}{2}}(-2x\,dx) \\
&= 2\pi R \int_{u=R^2}^{0} \left(\frac{-1}{2}\right) u^{-\frac{1}{2}}\,du \\
&= 2\pi R \left.\frac{\left(\frac{-1}{2}\right)u^{\frac{1}{2}}}{\frac{1}{2}}\right|_{R^2}^{0} \\
&= -2\pi R\left(0^{\frac{1}{2}} - \left(R^2\right)^{\frac{1}{2}}\right) \\
& 2\pi R^2
\end{aligned}$$

Recalling this is just the surface area of the top half we multiply by 2 to obtain the total surface area of the sphere.

$$SA = 4\pi R^2.$$

2. (15 points) Do **one** of the following.

(a) Molten glass is flowing through a rectangular opening of base width 5 m at a rate of 3 m/s. At time t s the depth of the glass is $H(t)$ m. The volume of glass passing through the opening in the time period from $t = a$ to $t = b$ is given by $\int_a^b 15H(t)\ dt$ cubic meters. Show that this is so by **carefully** building an appropriate Riemann sum that approximates the given volume. Keep track of units and explain all of your steps.

- **Solution:**
  i. First partition the time interval $[a, b]$ into n subintervals using the partition points $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$.
  ii. In the k'th subinterval (which has length Δt_k) we note the height of the flow of glass is approximately $H(t_k)$.
  iii. Since the opening (through which the glass is flowing) is 5 meters wide, then, during time interval Δt_k, a cross-section of the glass perpendicular to the flow has area approximately equal to
  $$5H(t_k)\ \text{m}^2$$
  iv. The glass is flowing at a rate of 3 $\frac{\text{m}}{\text{s}}$ so during time interval Δt_k there is 3 Δt_k m of flow passing through the opening.
  v. Thus, the volume of glass that passes the opening during time interval Δt_k is approximately
  $$\begin{aligned}\Delta V_k &= (5)\,H(t_k)\,(3\,\Delta t_k)\ \text{m}^3 \\ &= 15H(t_k)\ \Delta t_k\ \text{m}^3\end{aligned}$$
  vi. Since this formula works for any subinterval we have that the total volume of glass passing through the opening from time $t = a$ to $t = b$ is approximately
  $$V \approx \sum_{k=1}^{n} 15H(t_k)\ \Delta t_k\ \text{m}^3$$
  vii. Now taking the limit of this Riemann sum as $\Delta t_k \to 0$ we have
  $$\begin{aligned}V &= \lim_{||P|| \to 0} \sum_{k=1}^{n} 15H(t_k)\ \Delta t_k\ \text{m}^3 \\ &= \int_a^b 15H(t)\ dt.\end{aligned}$$

(b) Use an $\varepsilon - N$ argument to prove the following limit of the sequence $a_n = \frac{3n}{n+1}$ is correct.

$$\lim_{n\to\infty} \frac{3n}{n+1} = 3.$$

- **Solution:**
  i. Let ε be an arbitrary positve number and choose $N = \frac{3}{\varepsilon} - 1$.
  Then, whenever $n > N$ we have

$$\begin{aligned} n &> \frac{3}{\varepsilon} - 1 \\ n+1 &> \frac{3}{\varepsilon} \\ \varepsilon &> \frac{3}{n+1} \\ -\varepsilon &< \frac{-3}{n+1} \text{ and since } n \text{ and } \varepsilon \text{ are both positive} \\ -\varepsilon &< \frac{-3}{n+1} < \varepsilon \\ -\varepsilon &< \frac{3n - 3(n+1)}{n+1} < \varepsilon \\ -\varepsilon &< \frac{3n}{n+1} - 3 < \varepsilon \\ 3 - \varepsilon &< \frac{3n}{n+1} < 3 + \varepsilon \end{aligned}$$

  ii. This shows the definition of limit is satisfied so we conclude

$$\lim_{n\to\infty} \frac{3n}{n+1} = 3.$$

3. (20 points) When approximating the definite integral $\int_a^b f(x)\, dx$ by using n subintervals and either the Trapezoid Rule (T_n) or Simpson's Rule (S_n), the error bounds are given, respectively, by

$$|E_T| \le \frac{(b-a)^3}{12n^2} K \quad \text{and} \quad |E_S| \le \frac{(b-a)^5}{(180)\, n^4} M.$$

Here $|f''(x)| \le K$ and $\left|f^{(4)}(x)\right| \le M$ for all x in the interval $[a, b]$. Use this information to do **both** of the following.

(a) How large should n be to guarantee that the Trapezoid Rule approximation to $\int_0^1 e^{x^2}\, dx$ is accurate to within 0.00001? Justify your answer.

- **Solution:** Since $|f''(x)| = \left|2e^{x^2} + 4x^2 e^{x^2}\right| = 2e^{x^2} + 4x^2 e^{x^2}$ we first find the maximum value of $|f''(x)| = 2e^{x^2} + 4x^2 e^{x^2}$ on the interval $0 \le x \le 1$
  This can be done either by graphing or by noting that

$$\frac{d}{dx}\left[2e^{x^2} + 4x^2 e^{x^2}\right] = 12xe^{x^2} + 8x^3 e^{x^2}$$

  is always positive on the interval $0 \le x \le 1$ (and hence the function $|f''(x)|$ is increasing) In either case, we see the maximum occurs when $x = 1$ and the maximum value is $|f''(1)| = 2e^1 + 4e^1 = 6e \approx 16.310 = K$

Thus we see the error satisfies

$$|I - T_n| \le \frac{(b-a)^3}{12n^2} K \ \le \frac{(1-0)^3}{12n^2} 16.310$$

Inserting the desired level of accuracy and solving for n we have

$$\begin{aligned} \frac{(1-0)^3}{12n^2} 16.310 &\le 0.00001 \\ \frac{(1-0)^3}{12(0.00001)} 16.310 &\le n^2 \\ 368.67 &\le n \end{aligned}$$

- We conclude that a use of $n = 369$ will guarantee the Trapezoid rule accurate to within the desired tolerance.

(b) How large should n be to guarantee the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001? Justify your answer.

- **Solution:** Mimicing the process of part (a) we first find the maximum of $\left|f^{(4)}(x)\right| = \left|12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}\right| = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}$ on the interval $0 \le x \le 1$.
  This also occurs when $x = 1$ (again, either use a graph or note $\frac{d}{dx}\left[12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}\right] = 120xe^{x^2} + 160x^3e^{x^2} + 32x^5e^{x^2}$ is always positive on the given interval.)
  So the maximum value of $\left|f^{(4)}(x)\right|$ occurs at $x = 1$ and yields

$$\begin{aligned} \left|f^{(4)}(1)\right| &= 12e^1 + 48e^1 + 16e^1 \\ &= 76e \\ &\approx 206.59 = M \end{aligned}$$

Using this we solve the appropriate inequality for n

$$\begin{aligned} |I - S_n| &\le \frac{(b-a)^5}{(180)n^4} M < 0.00001 \\ \frac{(1-0)^5}{(180)n^4} 206.59 &< 0.00001 \\ \frac{(1-0)^5}{180(0.00001)} 206.59 &< n^4 \\ 18.406 &< n \end{aligned}$$

- Since we are using Simpson's rule n must be an even integer so we need $n = 20$ to guarantee the desired accuracy of our approximation.

Useful Information: If $f(x) = e^{x^2}$, then $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ and $f^{(4)}(x) = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}$.

4. (15 points each) Do any **two** of the following.

(a) In a population of 1000 people, one method of modelling the spread of a rumor is to use

$$\frac{dy}{dt} = ky(1000 - y).$$

Using the initial condition $y(0) = 1$, the value $k = 0.1$ and the fact that $\frac{1}{y(1000-y)} = \frac{1/1000}{y} + \frac{1/1000}{1000-y}$, solve this separable differential equation and determine $y(15)$.
[You don't need this information but, in case you are interested, t is time measured in days, $y(t)$ represents the number of people who know the rumor at time t, $1000 - y$ represents the number who don't, $y(1000 - y)$ represents the number of possible meetings of people where the rumor could be spread and $k = 0.1$ represents the proportion of these meetings in which the rumor is transferred.]

- **Solution:** Separating variables we have

$$\begin{aligned}
\frac{1}{y\left((1000-y)\right)}dy &= k\,dx \\
\int \frac{1}{y\left(1000-y\right)}dy &= \int k\,dx \\
\int \left(\frac{1/1000}{y} + \frac{1/1000}{1000-y}\right) dy &= \int k\,dx \\
1/1000 \int \frac{1}{y}\,dy + 1/1000 \int \frac{1}{1000-y}\,dy &= \int k\,dx \\
\frac{1}{1000}\ln|y| - \frac{1}{1000}\ln\left(1000-y\right) &= kx + C \\
\frac{1}{1000}\ln\left|\frac{y}{1000-y}\right| &= kx + C \\
\ln\left|\frac{y}{1000-y}\right| &= 1000\left(0.1\right)x + C_1 \\
e^{\ln\left|\frac{y}{1000-y}\right|} &= e^{1000(0.1)x}e^{C_1} \\
\frac{y}{1000-y} &= Me^{100x}
\end{aligned}$$

Using the initial condition that $y\left(0\right) = 1$ we have

$$\frac{1}{1000-1} = Me^{100(0)} = M$$

So

$$\begin{aligned}
\frac{y}{1000-y} &= \frac{1}{999}e^{100x} \\
y &= \frac{1}{999}e^{100x}\left(1000-y\right) \\
y &= \frac{1000}{999}e^{100x} - \frac{1}{999}e^{100x}y \\
y\left(1 + \frac{1}{999}e^{100x}\right) &= \frac{1000}{999}e^{100x} \\
y\left(x\right) &= \frac{\frac{1000}{999}e^{100x}}{1 + \frac{1}{999}e^{100x}}
\end{aligned}$$

Thus

$$y\left(15\right) = \frac{\frac{1000}{999}e^{(100)(15)}}{1 + \frac{1}{999}e^{(100)(15)}}$$

(b) The disk enclosed by the circle $x^2 + y^2 = 4$ is revolved about the y - axis to generate a solid ball. A hole of diameter 2 (radius 1) is then bored through the ball along the y-axis. Set up, but do not evaluate, definite integral(s) that give the volume of this “cored” solid ball.

- **Solution using the disk method:**
  i. The solid is obtained by rotating the region that is to the right of the line $x = 1$ and inside the circle $x^2 + y^2 = 4$ about the $y-$ axis.
  ii. The points where $x = 1$ intersects the circle are where $1^2 + y^2 = 4$.That is: $\left(1, \sqrt{3}\right)$, $\left(1, -\sqrt{3}\right)$ so we .
  iii. Cross sections perpendicular to the y- axis are washers with small radius $r = 1$ and large radius $x = \sqrt{4 - y^2}$.
  iv. Thus the volume is
$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\pi (1)^2 - \pi \left(\sqrt{4 - y^2}\right)^2\right) dy$$
- **Solution using the method of cylindrical shells**
  i. We rotate the same region about the y -axis just as in the solution via disk method.
  ii. The radius of our shells is just x
  iii. The height of the shell located at position x is determined by the two points where the vertical line through x meets the circle. That is, $\left(x, \sqrt{4 - x^2}\right)$and $\left(x, -\sqrt{4 - x^2}\right)$.
  iv. Thus, the volume is
$$\begin{aligned} V &= 2\pi \int_1^2 x \left(\sqrt{4 - x^2} - \left(-\sqrt{4 - x^2}\right)\right) dx \\ & 4\pi \int_1^2 x\sqrt{4 - x^2}\, dx. \end{aligned}$$

(c) A solid is generated by rotating about the x - axis the region bounded by the x - axis, the y - axis and the curve $y = f(x)$, where $f(x) \geq 0$ and $x \geq 0$. [That is, the graph of $y = f(x)$ lies in the first quadrant.] The function $f(x)$ has the property that the volume, $V(b)$, generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for every $b > 0$.Find the function $f(x)$.

i. **Solution:** Using the disk method we see the volume from $x = 0$ to $x = b$ is given by
$$b^2 = V(b) = \int_0^b \pi f^2(x)\, dx.$$
ii. Taking the derivative with respect to b and using a fundamental theorem of calculus we obtain
$$\begin{aligned} 2b &= V'(b) = \pi f^2(b) \\ f^2(b) &= \frac{2b}{\pi} \\ f(x) &= \sqrt{\frac{2x}{\pi}}. \end{aligned}$$

(d) The base of a solid sits on the region in the xy-plane bounded by the x-axis and the graph of the semicircle $y = \sqrt{4 - x^2}$. If cross sections perpendicular to the x-axis are rectangles with height twice as great as their base, what is the volume of the solid?

- **Solution:** The base of the rectangle that occurs at location x has length equal to the second coordinate of the point $\left(x, \sqrt{4 - x^2}\right)$.
  i. This means that $A(x) = \sqrt{4 - x^2}\left(2\sqrt{4 - x^2}\right) = 8 - 2x^2$

ii. The volume is thus

$$
\begin{aligned}
V &= \int_{-2}^{2} \left(8 - 2x^2\right) dx \\
&= 8x - \frac{2}{3}x^3 \Big|_{-2}^{2} \\
&= \frac{64}{3} \text{ units cubed}
\end{aligned}
$$

5. **Skip**

6. (20 points) Do **one** of the following.

   (a) A hemispherical water tank of radius 10 feet is being pumped out. [Recall that water weighs 62.4 pounds per cubic foot.]

   i. Show that a cross section y feet above the center of the tank is a disk of radius $\sqrt{100 - y^2}$ feet.

   A. **Solution:** We set the origin at the center of the sphere of which the tank is the top half.

   B. Then the tank is obtained by rotating the portin of the graph of $x^2 + y^2 = 100$ in the first quadrant about the y- axis.

   C. The horizontal line y units above the x - axis meets the graph of $x^2 + y^2 = 100$ in the first quadrant at point $\left(\sqrt{100 - y^2}, y\right)$.

   D. The first coordinate of this point is the radius of the disk cross-section at height y.

   ii. Compute the work done in lowering the water level from 2 feet below the top of the tank to 4 feet below the top of the tank given that the pump is placed 3 feet above the top of the tank.

   A. **Solution:** We have Δy as the thickness of the slab of water that is centered at height y.Then

$$
\begin{aligned}
\Delta V &= \pi \left(\sqrt{100 - y^2}\right)^2 \Delta y \text{ m}^3 \\
\Delta F &= (62.4) \frac{\text{lb}}{\text{ft}^3} \pi \left(\sqrt{100 - y^2}\right)^2 \Delta y \text{ m}^3 \\
&= (62.4)\ \pi \left(\sqrt{100 - y^2}\right)^2 \Delta y \text{ lb} \\
\Delta W &= (13 - y)(62.4)\ \pi \left(\sqrt{100 - y^2}\right)^2 \Delta y \text{ ft-lb} \\
W &= \int_6^8 (13 - y)(62.4)\ \pi \left(\sqrt{100 - y^2}\right)^2 dy \text{ ft-lb} \\
&= \int_6^8 (13 - y)(62.4)\ \pi \left(100 - y^2\right) dy \text{ ft-lb}
\end{aligned}
$$

   B. $\int_6^8 (13 - y)(62.4)\pi\left(100 - y^2\right) dy = 1.2102 \times 10^5$ ft-lb

   (b) Find the natural length of a heavy metal spring, given that the work done in stretching it from a length of 2 feet to length of 2.1 feet is one-half the work done in stretching it from a length of 2.1 feet to a length of 2.2 feet.

- **Solution:** We know that for springs, if a and b are distances from the natural length n then

$$\begin{aligned} W &= \int_a^b kx\,dx \\ &= \frac{1}{2}kx^2\Big|_a^b \\ &= \frac{1}{2}k\left(b^2-a^2\right). \end{aligned}$$

i. We are not told the natural length but if we designate that length by the letter n then we can write $2 = n + a$ and then the work done in stretching the spring from 2 feet to 2.1 feet is

$$\begin{aligned} W &= \int_a^{a+0.1} kx\,dx \\ &= \frac{1}{2}k\left((a+0.1)^2-a^2\right) \\ &= \frac{1}{2}k\,(0.2a+0.01) \end{aligned}$$

and the work done in stretching the spring from 2.1 feet to 2.2 feet is

$$\begin{aligned} \int_{a+0.1}^{a+0.2} kx\,dx &= \frac{1}{2}k\left((a+0.2)^2-(a+0.1)^2\right) \\ &= \frac{1}{2}k\,(0.2a+0.3) \end{aligned}$$

Since the first amount of work is $\frac{1}{2}$ the second amount of work we have

$$\begin{aligned} \frac{1}{2}k\,(0.2a+0.01) &= \left(\frac{1}{2}\right)\frac{1}{2}k\,(0.2a+0.03) \\ 0.2a+0.01 &= \frac{1}{2}\,(0.2a+0.03) \\ 0.2a+0.01 &= 0.1a+0.015 \\ 0.1a &= 0.005 \\ a &= 0.05 \end{aligned}$$

This means the natural length $n = 2 + a = 2.05$ feet.